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Poisson Summation Formula Associated with the Fractional Gabor Transform

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Abstract: **The application of linear canonical transform in quantum mechanics has focused attention on its complex extension. Fractional Gabor transform is special case of complex linear canonical transform. The present paper investigates the generalized Poisson summation formula for the fractional Gabor transform of the periodic functions of compact support. Then some results associated with this novel formula have been presented.**

Keywords: **fractional Fourier transform, linear canonical transform, Poisson summation formula, Gabor transform, fractional Gabor transform, testing function space.**

I. INTRODUCTION

Linear canonical transform is prominently used in optics [7], radar system analysis [8], signal processing [4] etc. In Fourier analysis Poisson summation formula is the relation that defines the periodic summation of a function in terms of a Fourier transform of discrete samples of the original function. The Poisson summation formula associated with the fractional Fourier transform was studied in [2].

Many transforms such as Fourier transform, fractional Fourier transform, Fresnel transform, Chirp transform are all special cases of linear canonical transform. Many of its properties are found in the literature. The properties and application of the sampling formulae in the traditional Fourier domain have been studied and sampling signal analysis have been extended for band limited signals in fractional Fourier domain in the literature. The simple form of fractional Gabor transform of signal $f(x)$ with rotation α is defined as in [5]. A generalized Poisson summation formula and its application to fast linear convolution have been studied in [3]. The Poisson summation formula associated with fractional Laplce transform derived in [1]. The Poisson summation formula associated with the generalized fractional Hilbert transform had been studied in [6] and generalization of the Poisson summation formula associated with the linear canonical transform was studied in [9]. The objective of this paper is to study the Poisson summation formula in fractional Gabor transform domain.

This paper is organized as follows. Preliminaries are presented in section 2. The Poisson summation formula for generalized fractional Gabor transform is obtained in fractional domain in section 3. Some important properties of Poisson summation formula are given in section 4. Section 5 concludes the paper.

II. PRELIMINARIES

2.1 GENERALIZED FRACTIONAL GABOR TRANSFORM:

The generalized fractional Gabor transform of $f(x) \in E'(R^n)$, where $E'(R^n)$ is dual of the testing function space $E(R^n)$, can be defined as

$$[G_\alpha f(x)](u) = \langle f(x), K_\alpha(x,u,t) \rangle \text{ For each } u \in R \quad (1)$$

Where, $K_\alpha(x,u,t)=\sqrt{\frac{1-i\cot\alpha}{2\pi}}e^{i\frac{(x^2+u^2)\cot\alpha}{2}}e^{\frac{-(x-t)^2\csc\alpha}{2}}e^{-iux\csc\alpha}$

The right hand side of equation (1) has meaning as the application of $f(x)\in E'$ to $K_\alpha(x,u,t)\in E$. $[G_\alpha f(x)](u)$ is α^{th} order generalized fractional Gabor transform of the function $f(x)$.

2.2 POISSON SUMMATION FORMULA IN THE FOURIER DOMAIN:

The Poisson summation formula demonstrates that the sum of infinite samples in time domain of a function $f(x)$ is equivalent to sum of infinite samples of $F(\omega)$ in Fourier domain. This can be represented as follows,

$$\sum_{k=-\infty}^{\infty} f(x+k\tau)=\frac{1}{\tau}\sum_{n=-\infty}^{\infty}F\left(\frac{n}{\tau}\right)e^{\frac{inx}{\tau}} \tag{2}$$

$$\sum_{k=-\infty}^{\infty} f(k\tau)=\frac{1}{\tau}\sum_{n=-\infty}^{\infty}F\left(\frac{n}{\tau}\right)$$

Where $F(\omega)$ is the Fourier transform of the function $f(x)$.

2.3 FUNCTION WITH COMPACT SUPPORT IN THE FOURIER DOMAIN:

Let $F(\omega)$ be the Fourier transform of the function $f(x)$. Then $f(x)$ is said to have compact support in the Fourier domain if $F(\omega)=0$ for $|\omega|>\Omega$. Where Ω is real number.

2.4 GAUSSION PERIODICFUNCTION:

The function $f(x)$ is said to be Gaussian Periodic with period τ and order α if

$$e^{\frac{x^2}{2}\cot\alpha} f(x)=e^{\frac{(x+\tau)^2}{2}\cot\alpha} f(x+\tau) \tag{3}$$

III. GENERALIZED POISSON SUMMATION FORMULA

3.1 FUNCTIONS WITH COMPACT SUPPORT IN GENERALIZED FRACTIONAL GABOR TRANSFORM DOMAIN:

Let $G_\alpha(\omega)$ be the fractional Gabor transform of the function $f(x)$. Then $f(x)$ is said to have compact support in the Fractional Gabor domain if $G_\alpha(\omega)=0$ for $|\omega|>\Omega_\alpha$ .where Ω_α is some real number.

3.2 POISSON SUMMATION FORMULA FOR GENERALIZED FRACTIONAL GABOR TRANSFORM:

If $f(x)$ is the function such that its Gabor transform has compact support say Ω_α in generalized fractional Gabor transform domain for $|u|>\Omega_\alpha$ then

$$\sum_{k=-\infty}^{\infty} f(x+k\tau)e^{\frac{i(x+k\tau)^2}{2}\cot\alpha}e^{\frac{-(x+k\tau-t)^2}{2}\csc\alpha}=\frac{c}{\tau}\sum_{n=-\infty}^{\infty}e^{\frac{-in^2\sin 2\alpha}{4\tau^2}}[G_\alpha f(x)]\left(\frac{n\sin\alpha}{\tau}\right)e^{\frac{inx}{\tau}}.$$

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Proof: By definition of fractional Gabor transform

$$[G_\alpha f(x)](u)=\sqrt{\frac{1-i\cot\alpha}{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i\frac{(x^2+u^2)\cot\alpha}{2}}e^{\frac{-(x-t)^2\csc\alpha}{2}}e^{-iux\csc\alpha}dx$$

$$\sqrt{\frac{2\pi}{1-i\cot\alpha}}e^{\frac{-iu^2\cot\alpha}{2}}G_\alpha(u)=\int_{-\infty}^{\infty}\left\{f(x)e^{i\frac{x^2\cot\alpha}{2}}e^{\frac{-(x-t)^2\csc\alpha}{2}}\right\}e^{-i(u\csc\alpha)x}dx$$

$$=G\{g(x)\}(u\csc\alpha)=G(v)$$

(4)

Where $g(x)=f(x)e^{i\frac{x^2\cot\alpha}{2}}e^{\frac{-(x-t)^2\csc\alpha}{2}}$

And $v=u\csc\alpha$

(5)

Using the Poisson summation formula for a function $g(x)$ in the Fourier domain

$$\sum_{k=-\infty}^{\infty}g(x+k\tau)=\frac{1}{\tau}\sum_{n=-\infty}^{\infty}G\left(\frac{n}{\tau}\right)e^{\frac{inx}{\tau}}$$

Using equation (4) in above equation, we get

$$=\frac{1}{\tau}\sum_{n=-\infty}^{\infty}G(v)e^{\frac{inx}{\tau}} \qquad (6)$$

Where $v=\frac{n}{\tau}$

Using equation (4) in equation (6)

$$\sum_{k=-\infty}^{\infty}g(x+k\tau)=\frac{1}{\tau}\sum_{n=-\infty}^{\infty}\sqrt{\frac{2\pi}{1-i\cot\alpha}}e^{\frac{-iu^2\cot\alpha}{2}}[G_\alpha f(x)](u)e^{\frac{inx}{\tau}}$$

$$=\frac{c}{\tau}\sum_{n=-\infty}^{\infty}e^{\frac{-i(v\sin\alpha)^2\cot\alpha}{2}}[G_\alpha f(x)](v\sin\alpha)e^{\frac{inx}{\tau}}$$ Using equation (4)

Where $c=\sqrt{\frac{2\pi}{1-i\cot\alpha}}$

$$=\frac{c}{\tau}\sum_{n=-\infty}^{\infty}e^{\frac{-iv^2\sin^2\alpha\cdot\cot\alpha}{2}}[G_\alpha f(x)](v\sin\alpha)e^{\frac{inx}{\tau}}$$

$$=\frac{c}{\tau}\sum_{n=-\infty}^{\infty}e^{\frac{-in^2\sin 2\alpha\cdot}{4\tau^2}}[G_\alpha f(x)]\left(\frac{n}{\tau}\sin\alpha\right)e^{\frac{inx}{\tau}}$$

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Using equation (5)

$$\sum_{k=-\infty}^{\infty} f(x+k\tau) e^{\frac{i(x+k\tau)^2}{2}\cot\alpha} e^{\frac{-(x+k\tau-t)^2}{2}\csc\alpha} = \frac{c}{\tau}\sum_{n=-\infty}^{\infty} e^{\frac{-in^2\sin 2\alpha}{4\tau^2}} \left[G_\alpha f(x)\right]\left(\frac{n\sin\alpha}{\tau}\right) e^{\frac{inx}{\tau}} \qquad (7)$$

3.3 POISSON SUMMATION FORMULAS FOR FRACTIONAL GABOR TRANSFORM WHEN $x = 0$:

Put $x = 0$ in equation (7), we get

$$\sum_{k=-\infty}^{\infty} f(k\tau) e^{\frac{i(k\tau)^2}{2}\cot\alpha} e^{\frac{-(k\tau-t)^2}{2}\csc\alpha} = \frac{c}{\tau}\sum_{n=-\infty}^{\infty} e^{\frac{-in^2\sin 2\alpha}{4\tau^2}} \left[G_\alpha f(0)\right]\left(\frac{n\sin\alpha}{\tau}\right) \qquad (8)$$

Equation (7) and (8) can be seen as Poisson summation formula associated with fractional Gabor transform of order α . It is clear from the above equations that the infinite sum of the periodic replica of function $f(x)$ is equal to infinite sum of its fractional Gabor transform.

Next we prove four corollaries which give reduce form of Poisson summation formula for special cases of the variable in the fractional Gabor transform.

3.4 COROLLARY:

If $f(x)$ is the function such that its Gabor transform has compact support say Ω_α in generalized fractional Gabor transform domain and $\frac{\sin\alpha}{\tau} > \Omega_\alpha$ then

$$\sum_{k=-\infty}^{\infty} f(x+k\tau) e^{\frac{i(x+k\tau)^2}{2}\cot\alpha} e^{\frac{-(x+k\tau-t)^2}{2}\csc\alpha} = \frac{c}{\tau} G_\alpha(0)$$

Proof: If $f(x)$ is the function such that its Gabor transform has compact support say Ω_α in generalized fractional Gabor transform domain and $\frac{\sin\alpha}{\tau} > \Omega_\alpha$ then $G_\alpha\left(\frac{n\sin\alpha}{\tau}\right) = 0$ when $n \neq 0$.

From equation (7)

$$\sum_{k=-\infty}^{\infty} f(x+k\tau) e^{\frac{i(x+k\tau)^2}{2}\cot\alpha} e^{\frac{-(x+k\tau-t)^2}{2}\csc\alpha} = \frac{c}{\tau}\sum_{n=-\infty}^{\infty} e^{\frac{-in^2\sin 2\alpha}{4\tau^2}} \left[G_\alpha f(x)\right]\left(\frac{n\sin\alpha}{\tau}\right) e^{\frac{inx}{\tau}}$$

Given $\frac{\sin\alpha}{\tau} > \Omega_\alpha$

$\therefore \left|\frac{n\sin\alpha}{\tau}\right| > |n\Omega_\alpha| > \Omega_\alpha$ for all n from $-\infty$ to ∞ except $n = 0$.

$\therefore$ Right hand side exists only for $n = 0$ and hence,

$$\sum_{k=-\infty}^{\infty} f(x+k\tau) e^{\frac{i(x+k\tau)^2}{2}\cot\alpha} e^{\frac{-(x+k\tau-t)^2}{2}\csc\alpha} = \frac{c}{\tau} G_\alpha(0).$$

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3.5 COROLLARY:

If $f(x)$ is the function such that its Gabor transform has compact support say Ω_α in generalized fractional Gabor transform domain and as $\frac{\Omega_\alpha}{2} < \frac{\sin\alpha}{\tau} < \Omega_\alpha$ then the Poisson summation formula reduces to,

$$\sum_{k=-\infty}^{\infty} f(x+k\tau)e^{\frac{i(x+k\tau)^2}{2}\cot\alpha} e^{\frac{-(x+k\tau-t)^2}{2}\csc\alpha} = \frac{c}{\tau}\left\{[G_\alpha f(x)](0) + e^{\frac{-i\sin 2\alpha}{4\tau^2}}\left[[G_\alpha f(x)]\left(\frac{\sin\alpha}{\tau}\right)e^{\frac{ix}{\tau}} + [G_\alpha f(x)]\left(\frac{-\sin\alpha}{\tau}\right)e^{\frac{-ix}{\tau}}\right]\right\}$$

Proof: Given that $\frac{\Omega_\alpha}{2} < \frac{\sin\alpha}{\tau} < \Omega_\alpha$

$\therefore \left|\frac{\sin\alpha}{\tau}\right| > \left|\frac{\Omega_\alpha}{2}\right|$ And $\left|\frac{\sin\alpha}{\tau}\right| < \Omega_\alpha$

Only for $n = 0, 1$ and -1, we get $\left|\frac{n\sin\alpha}{\tau}\right| < \Omega_\alpha$

Otherwise for all other values of n, $\left|\frac{n\sin\alpha}{\tau}\right| > \Omega_\alpha$

$\therefore$ Right hand side summation will contain only three nonzero terms for $n = 0, 1$ and -1.

From equation (7)

$$\sum_{k=-\infty}^{\infty} f(x+k\tau)e^{\frac{i(x+k\tau)^2}{2}\cot\alpha} e^{\frac{-(x+k\tau-t)^2}{2}\csc\alpha}$$

$$= \frac{c}{\tau}\left\{[G_\alpha f(x)](0) + [G_\alpha f(x)]\left(\frac{\sin\alpha}{\tau}\right)e^{\frac{-i\sin 2\alpha}{4\tau^2}} e^{\frac{ix}{\tau}} + [G_\alpha f(x)]\left(\frac{-\sin\alpha}{\tau}\right)e^{\frac{-i\sin 2\alpha}{4\tau^2}} e^{\frac{-ix}{\tau}}\right\}$$

$$= \frac{c}{\tau}\left\{[G_\alpha f(x)](0) + e^{\frac{-i\sin 2\alpha}{4\tau^2}}\left[[G_\alpha f(x)]\left(\frac{\sin\alpha}{\tau}\right)e^{\frac{ix}{\tau}} + [G_\alpha f(x)]\left(\frac{-\sin\alpha}{\tau}\right)e^{\frac{-ix}{\tau}}\right]\right\}.$$

3.6 COROLLARY:

If $f(x)$ is the function such that its Gabor transform has compact support say Ω_α in generalized fractional Gabor transform domain and as $\frac{\Omega_\alpha}{n} < \frac{\sin\alpha}{\tau} < \frac{\Omega_\alpha}{n-1}$ then the Poisson summation formula reduces to,

$$\sum_{k=-\infty}^{\infty} f(x+k\tau)e^{\frac{i(x+k\tau)^2}{2}\cot\alpha} e^{\frac{-(x+k\tau-t)^2}{2}\csc\alpha}$$

$$= \frac{c}{\tau}\left\{[G_\alpha f(x)](0) + \sum_{k=1}^{n} e^{\frac{-ik^2\sin 2\alpha}{4\tau^2}}\left[[G_\alpha f(x)]\left(\frac{k\sin\alpha}{\tau}\right)e^{\frac{ikx}{\tau}} + [G_\alpha f(x)]\left(\frac{-k\sin\alpha}{\tau}\right)e^{\frac{-ikx}{\tau}}\right]\right\}$$

Proof: It is clear when $\frac{\Omega_\alpha}{n} < \frac{\sin\alpha}{\tau} < \frac{\Omega_\alpha}{n-1}$, on right hand side only $(2n+1)$ (that is from $-n$ to n terms will have nonzero values, and all other terms will vanish, thus we get

$$\sum_{k=-\infty}^{\infty} f(x+k\tau) e^{\frac{i(x+k\tau)^2}{2}\cot\alpha} e^{\frac{-(x+k\tau-t)^2}{2}\csc\alpha}$$

$$=\frac{c}{\tau}\left\{[G_\alpha f(x)](0)+\sum_{k=1}^{n} e^{\frac{-ik^2\sin 2\alpha}{4\tau^2}}\left[[G_\alpha f(x)]\left(\frac{k\sin\alpha}{\tau}\right)e^{\frac{ikx}{\tau}}+[G_\alpha f(x)]\left(\frac{-k\sin\alpha}{\tau}\right)e^{\frac{-ikx}{\tau}}\right]\right\}.$$

3.7 COROLLARY:

If $f(x)$ is the function such that its Gabor transform has compact support say Ω_α in generalized fractional Gabor transform domain then

$$\sum_{k=-\infty}^{\infty} f(x+k\tau) e^{\frac{i(x+k\tau)^2}{2}\cot\alpha} e^{\frac{-(x+k\tau-t)^2}{2}\csc\alpha}$$

$$=\begin{cases} \frac{c}{\tau}[G_\alpha f](0), & \frac{\sin\alpha}{\tau} > \Omega_\alpha \\ \frac{c}{\tau}\left\{[G_\alpha f](0)+\sum_{k=1}^{n} e^{\frac{-i\sin 2\alpha}{4\tau^2}}\left[[G_\alpha f]\left(\frac{\sin\alpha}{\tau}\right)+[G_\alpha f]\left(\frac{-\sin\alpha}{\tau}\right)\right]\right\}, & \frac{\Omega_\alpha}{2} < \frac{\sin\alpha}{\tau} < \Omega_\alpha \\ \frac{c}{\tau}\left\{[G_\alpha f](0)+\sum_{k=1}^{n} e^{\frac{-ik^2\sin 2\alpha}{4\tau^2}}\left[[G_\alpha f]\left(\frac{k\sin\alpha}{\tau}\right)+[G_\alpha f]\left(\frac{-\sin\alpha}{\tau}\right)\right]\right\}, & \frac{\Omega_\alpha}{n} < \frac{\sin\alpha}{\tau} < \frac{\Omega_\alpha}{n-1} \end{cases}$$

Proof: This result can be obtained easily by letting $t=0$ in result 3.4, 3.5 and 3.6.

IV. PROPERTY

In order to device some properties of infinite sum of fractional Gabor transform which can be obtained from the function $\sum_{k=-\infty}^{\infty} f(x+k\tau) e^{\frac{i(x+k\tau)^2}{2}\cot\alpha} e^{\frac{-(x+k\tau-t)^2}{2}\csc\alpha}$, We treat this as a special function of x say,

$$y(x)=\sum_{k=-\infty}^{\infty} f(x+k\tau) e^{\frac{i(x+k\tau)^2}{2}\cot\alpha} e^{\frac{-(x+k\tau-t)^2}{2}\csc\alpha}$$

(9)

Then $y(x)$ is Gaussian periodic function with period τ .

Proof: Consider,

$$y(x+\tau)=\sum_{k=-\infty}^{\infty} f(x+\tau+k\tau) e^{\frac{i(x+\tau+k\tau)^2}{2}\cot\alpha} e^{\frac{-(x+\tau+k\tau-t)^2}{2}\csc\alpha} = y(x) .$$

Hence $y(x)$ is Gaussian periodic.

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V. CONCLUSION

In this paper, generalized Poisson summation formula has been proposed for fractional Gabor transform in fractional Gabor transform domain. Some novel results associated with Poisson summation formula have been derived in the form of corollaries. $y(x)$ is Gaussian periodic function is obtained in terms of property.

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